MATH2050C Selected Solution to Assignment 9

Section 4.3

(3) Let $M > 0$ be given. We take $\delta = 1/M^2$. Then for $x, 0 < |x| < \delta$,

$$\frac{1}{\sqrt{x}} > \frac{1}{\sqrt{\delta}} = M \ ,$$

so $\lim_{x\to 0^+} 1/\sqrt{|x|} = \infty$.

(5a) The right hand limit is ∞. (More precisely, the right hand limit does not exist; it diverges to ∞.) Let $M > 0$. Choose $\delta = 1/M$. Then for $x, 1 < x < 1 + 1/M$,

$$\frac{x}{x-1} \geq \frac{1}{x-1} > M \ .$$

(5b) The limit does not exist. The right limit diverges to ∞ and the left limit diverges to $-\infty$. For $M > 0$. Choose $\delta = \min\{1/2, 1/(2M)\}$. Then for $x, 0 < x - 1 < \delta$, we have $x > 1 - \delta \geq 1 - 1/2 = 1/2$. Thus,

$$\frac{x}{x-1} \geq \frac{1/2}{x-1} > \frac{1}{2\delta} \geq \frac{1}{2 \times M/2} = M \ ,$$

which shows that $\lim_{x\to 1^+} x/(x-1) = \infty$. Similarly, one can show that $\lim_{x\to 1^-} x/(x-1) = -\infty$.

(5h) The limit exists and is equal to -1. For $x > 0$,

$$0 \leq \left| \frac{\sqrt{x} - x}{\sqrt{x} + x} - (-1) \right| = \left| \frac{2\sqrt{x}}{\sqrt{x} + x} \right| \leq \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}} \ .$$

By Squeeze Theorem,

$$0 \leq \lim_{x\to\infty} \frac{\sqrt{x} - x}{\sqrt{x} + x} \leq \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0 \ .$$

Supplementary Problems

Justify your answers in the following problems.

1. Evaluate

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2}.$$

Solution. Use $(x^2 - 4)/(x - 2) = (x - 2)(x + 2)/(x - 2) = x + 2$ for $x \neq 2$, we have

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} (x + 2) = 4.$$

2. Evaluate
$$\lim_{x\to -3}\frac{x^2-2x-15}{x+3}.$$

Solution. Use $x^2-2x-15=(x+3)(x-5)$, we have $\frac{x^2-2x-15}{x+3}=\frac{(x+3)(x-5)}{x+3}=x-5$ as long as $x+3\neq 0$. Therefore,
$$\lim_{x\to -3}\frac{x^2-2x-15}{x+3}=\lim_{x\to -3}(x-5)=-8.$$

3. Evaluate
$$\lim_{x\to\infty}\frac{\cos 1/x}{x}.$$

Solution. Observe that $|\cos 1/x|\le 1$ for all x. For $\varepsilon>0$, take $K=1/\varepsilon$. Then
$$\left|\frac{\cos 1/x}{x}\right|\le\frac{1}{x}<\varepsilon\ ,\quad \forall x>K.$$
We conclude that the limit is equal to 0.

4. Find $\lim_{x\to 0^+}\dfrac{5x-\sqrt{x}}{\sqrt{x}-x^3}$ for $c=0^+$ and ∞.

Solution (a)
$$\lim_{x\to 0^+}\frac{5x-\sqrt{x}}{\sqrt{x}-x^3}=\lim_{x\to 0^+}\frac{5\sqrt{x}-1}{1-x^{5/2}}=\frac{\lim_{x\to 0^+}(5\sqrt{x}-1)}{\lim_{x\to 0^+}(1-x^{5/2})}=-1\ .$$

(b) For $x\ge 1$,
$$\left|\frac{5x-\sqrt{x}}{\sqrt{x}-x^3}\right|=\left|\frac{5\sqrt{x}-1}{1-x^{5/2}}\right|\le\frac{5\sqrt{x}}{x^{5/2}-1}=\frac{5}{x^2}\frac{1}{1-x^{-5/2}}\ .$$
As
$$\lim_{x\to\infty}\frac{5}{x^2}\frac{1}{1-x^{-5/2}}=0\ ,$$
by Squeeze Theorem
$$\lim_{x\to\infty}\frac{5x-\sqrt{x}}{\sqrt{x}-x^3}=0\ .$$

5. Find $\lim_{x\to c}x^3e^{1/x}$ where $c=0^+,0^-,\infty$, and $-\infty$.

Solution (a) From $e^x=\sum_{k=0}^{\infty}x^k/k!$ we get $e^x\ge x^4/4!$ for $x\ge 0$. Therefore, $x^3e^{1/x}\ge x^3(1/x)^4/24=1/24x$ and
$$\lim_{x\to 0^+}x^3e^{1/x}\ge\frac{1}{24}\lim_{x\to 0^+}\frac{1}{x}=\infty\ .$$

(b) For $x<0$, $e^{1/x}=\frac{1}{e^{1/|x|}}\le 1$. Therefore,
$$|x^3e^{1/x}|\le|x|^3\to 0\ ,$$

as $x \to 0^-$.

(c) Using $e^x \geq 1$ for $x \geq 0$, $x^3 e^{1/x} \geq x^3$, hence

$$\lim_{x\to\infty} x^3 e^{1/x} = \lim_{x\to\infty} x^3 = \infty \ .$$

(d) Changing x to $-x$, this limit is equal to $-\lim_{x\to\infty} x^3 e^{-1/x}$. As $e^{-1/x}$ approaches $e^0 = 1$ as $x \to \infty$. We have $\lim_{x\to\infty} -x^3 e^{-1/x} = -\lim_{x\to\infty} x^3 \lim_{x\to\infty} e^{-1/x} = -\infty$.